LESSON 10 - STUDY GUIDE

Abstract. This lesson will be totally dedicated to the concept of convolution product between two functions. We are finally moving from studying functions in L^p spaces, a subject which sits within the boundaries of measure theory and functional analysis, to a topic which already is right at the core of harmonic analysis.

1. **Convolution.**

<u>**Study material:**</u> Sections **8.1 Preliminaries** and **8.2 Convolution** from chapter **8 Elements of Fourier Analysis** in [1], pgs. 235-246. Actually we will essentially focus on section 8.2, but if you are not familiar with the multi-index notation in $\mathbb{R}^n$ for sums, derivatives, powers, etc., C^∞ functions with compact support and Schwartz functions, then you should definitely take a closer look at section 8.1 where those concepts are covered and also try some of the exercises there, to get used to them. I will assume familiarity with multi-index notation and, at least, with C_c^∞ functions. Section **1.2 Convolution and Approximate Identities** in [2] is very similar and complete as well, but I will follow [1] more closely.

For f, g measurable functions defined on $\mathbb{R}^n$, their **convolution product** is the integral

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy, \tag{1.1}$$

whenever the integral exists. Obvious conditions for this to happen for all $x \in \mathbb{R}^n$ are, for example, when one of the functions is in $L^\infty(\mathbb{R}^n)$ and the other in $L^1(\mathbb{R}^n)$, or one of them is bounded with compact support while the other is continuous.

Intuitively, the convolution is an operator that integrates the translates of the function f, with a measure $g(y)dy$, weighted by the function g.

Observe that, besides the Lebesgue measure required to perform the integral, the only other concept that is used in this definition is the translation of a function, which of course is connected to the vector sum operation in $\mathbb{R}^n$. As I pointed out in this course's first lesson, describing what Harmonic Analysis is, from an abstract point of view, it boils down, at its essence, to the study of functions in locally compact abelian groups, where a measure which is invariant with respect to the group operation - the so called Haar measure - can always be defined uniquely, modulo a multiplicative constant. The convolution product is thus an operator that encapsulates these two fundamental concepts of harmonic analysis: in $\mathbb{R}^n$ the abelian group operation is the vector sum, thus the translations; and the Haar measure, invariant under such translations, is the Lebesgue measure. In terms of harmonic analysis of functions defined on $\mathbb{R}^n$, the other vector space operation in the functions' domains, i.e. multiplication of vectors $x \in \mathbb{R}^n$ by scalars, is an operation almost absent from the whole theory and in no way is it comparable, in terms of importance, to the vector sum, which is an absolutely vital ingredient. It therefore goes without saying that the definition of convolution product of two functions generalizes in a totally analogous way to any locally compact abelian group with its Haar measure (this more abstract approach is actually the way the definition is presented in [2], while [1] sticks to $\mathbb{R}^n$). And, in fact, can even be defined between more general objects than functions, such as measures and even distributions.

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To start with, and to be completely rigorous, we need to make sure that , if f and g are Lebesgue measurable on $\mathbb{R}^n$, so is $f(x-y)g(y)$. This, though, is not entirely trivial. I don't want to dwell too long on this technical issue, but suffices to say that Borel measurability follows easily if f is Borel measurable, as $f(x-y)$ is then the composition of a Borel measurable function with the continuous function $x-y : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and is thus Borel measurable. But even when f is Lebesgue measurable, the measurability of $f(x-y)g(y)$ does hold. See the discussion at the top of page 240 and Exercise 5, in page 245, in [1].

The first result summarizes the basic properties of the convolution.

Theorem 1.1. *Assuming that all integrals in question exist, we have*

1. $f * g = g * f$
2. $(f * g) * h = f * (g * h)$
3. $(\alpha f + \beta g) * h = (\alpha f) * h + (\beta g) * h = \alpha(f * h) + \beta(g * h)$, *for* $\alpha, \beta \in \mathbb{C}$
4. *For* $z \in \mathbb{R}^n$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$, *where* $\tau_z f(x) = f(x-z)$, *for* $x, z \in \mathbb{R}^n$
5. $\text{supp}\,(f * g) \subset \overline{\{x + y : x \in \text{supp}\ f, y \in \text{supp}\ g\}}$

Proof. The proof is very simple and should be attempted as an exercise. In any case, you can see it in [1], Proposition 8.6, pg. 240. □

An interesting way to see what the convolution does is to look at it from a perspective analogous to what is done in the theory of distributions, by analyzing its effect on test functions. The revolutionary idea of Laurent Schwartz, that led to the theory of distributions, was to shift the focus from the pointwise values of functions to the way they act "against" test functions by integration. So, imagine that one wants to study the properties of a function $F : \mathbb{R}^n \to \mathbb{C}$ which is, say, in $L^1_{loc}(\mathbb{R}^n)$ - this means that the function is locally in L^1, i.e. L^1 on any compact subset of $\mathbb{R}^n$. Notice that L^1_{loc} contains many, many types of functions: all continuous functions on $\mathbb{R}^n$ are in $L^1_{loc}(\mathbb{R}^n)$, and so are all $L^p(\mathbb{R}^n)$ functions, for any $1 \le p \le \infty$ (recall the property that, for sets of finite measure, which is the case with compacts in $\mathbb{R}^n$, one has $L^p \subset L^1$). Then, the idea is, rather than looking at the particular values $F(x)$ for all $x \in \mathbb{R}^n$ - which actually do not even matter on subsets of zero measure - to look instead at the values of

$$\int_{\mathbb{R}^n} F(x)\phi(x)dx,$$

for *all* "test functions" which are smooth, of compact support, $\phi \in C_c^\infty(\mathbb{R}^n)$. The fact that F is Lebesgue integrable over compacts, and that ϕ has compact support, guarantees that these integrals are always well defined. The important property, that makes this change of perspective work, is that no two different functions can have the same effect on *all* test functions. Or, to put it more precisely, if $F, G \in L^1_{loc}(\mathbb{R}^n)$ are such that

$$\int_{\mathbb{R}^n} F(x)\phi(x)dx = \int_{\mathbb{R}^n} G(x)\phi(x)dx, \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{R}^n),$$

then $F(x) = G(x)$ a.e. in $\mathbb{R}^n$ (try proving it... it is not a trivial exercise and requires a fine use of measure theory). In other words, functions can be fully identified and studied by their effect on all test functions, as a replacement for their pointwise values.

Let us then get back to the convolution and look at it from this perspective. Instead of the actual values of $f * g(x)$ we will see what it does to test functions (assuming all integrations hold). Then

$$\int_{\mathbb{R}^n} f * g(x)\phi(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)\phi(x)dydx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\phi(x+y)dydx,$$

by a simple change of variables. It all now becomes a bit more transparent: the convolution $f * g$ acts on a test function ϕ at the point $x + y$, where $x \in \text{supp}\, f$ and $y \in \text{supp}\, g$. Property (5) above becomes totally obvious, this way, and the commutativity of the convolution, Property (1), can simply be seen as a consequence of the commutativity of the group operation, the vector sum in the $\mathbb{R}^n$ case.

Now, from properties (3) and (4) in Theorem 1.1 above, we can also conclude that, for fixed g, say, $f*g$ is a linear operator in f, which commutes with translations, i.e. the operator applied to the translation of f is the same as the translation of the operator applied to f:

$$\tau_z(f*g) = (\tau_z f)*g.$$

Convolutions show up over and over again in harmonic analysis, whenever one studies linear operators that commute with translations. This permanent presence in the landscape of the theory is, of course, no coincidence, because the following characterization of such operators between L^p spaces can be proved.

Theorem 1.2. *Suppose $1 \le p, q \le \infty$. Suppose that T is a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ that commutes with translations. Then, there exists a unique (tempered) distribution μ such that*

$$T(f) = f*\mu, \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decaying functions in $\mathbb{R}^n$.

For those of you that already know a little bit about distributions, I suggest taking a look at section **2.5.1. Operators That Commute With Translations** in Grafakos' book [2], where this theorem and its proof can be found. For everyone else, just keep in mind that, essentially, every linear bounded operator between L^p spaces that commutes with translations in $\mathbb{R}^n$ is given by a convolution with a fixed function, or by a slightly more general creature: a distribution. This is the reason why convolution operators are so central in harmonic analysis, and indeed appear so frequently.

Now, besides very obvious specific cases where it is immediately clear that the convolution integral is well defined, we should start by studying more general families of functions for which the integral (1.1) exists, at least a.e. $x \in \mathbb{R}^n$. I mentioned in the previous lesson how a large part of the modern methods in harmonic analysis actually consist of trying to prove that the convolution integral is well defined, as well as trying to find estimates for it, as the functions f and g become more badly behaved, and eventually singular. And that is precisely the central subject of study of singular integral operators and Calderon-Zygmund theory. We will, however, start with the most basic and natural families of functions, with respect to Lebesgue integration: the L^p spaces.

We start with the case where $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. In this case, as mentioned in the previous lesson, we can think of the translates of f as a kernel of an integral operator, $K(x,y) = f(x-y)$, so that applying the Schur-Young theorem, we immediately obtain the following.

Proposition 1.3. (Young's Inequality) *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, with $1 \le p \le \infty$. Then, the convolution $f*g(x)$ given by the integral (1.1) is defined a.e. $x \in \mathbb{R}^n$, $f*g \in L^p(\mathbb{R}^n)$ and*

$$\|f*g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}.$$

Proof. One can simply apply the Schur-Young criterion and theorem to $K(x,y) = f(x-y)$, or instead just use the Minkowski integral inequality directly on $f*g$ to obtain

$$\|f*g\|_{L^p(\mathbb{R}^n)} = \|\int_{\mathbb{R}^n} f(\cdot - y)g(y)dy\|_{L^p(\mathbb{R}^n)} = \|\int_{\mathbb{R}^n} g(\cdot - y)f(y)dy\|_{L^p(\mathbb{R}^n)} \le$$

$$\int_{\mathbb{R}^n} \|g(\cdot - y)f(y)\|_{L^p(\mathbb{R}^n)}dy = \int_{\mathbb{R}^n} \|g(\cdot)\|_{L^p(\mathbb{R}^n)}|f(y)|dy = \|g\|_{L^p(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}.$$

□

In particular, we conclude from this result that the convolution of two functions in $L^1(\mathbb{R}^n)$ is defined a.e. and is an $L^1(\mathbb{R}^n)$ function as well. Therefore the operation $(f,g) \in L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \mapsto f*g \in L^1(\mathbb{R}^n)$ is indeed a product that turns $L^1(\mathbb{R}^n)$ into a commutative and associative algebra, a so called commutative Banach algebra (a Banach algebra is an associative algebra that is also a Banach space).

Another particular, and obvious case, of the previous result is when $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$. We will now consider the situation in which g is actually, not only bounded, but continuous or even differentiable. If one regards the convolution as a "continuous superposition" of translates of f, summed/integrated by the weighted measure $g(y)dy$ then it is expected that the convolution should inherit the same regularity as f. But as $f * g = g * f$, the roles of f and g can be exchanged and the same could be said about the regularity of g. Therefore, it should be true that if f or g are k-times differentiable, then for any multi-index $|\alpha| \leq k$ one should have

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g).$$

Moreover, the total differentiability of $f*g$ should result from the sums of the maximum differentiabilities of f and g separately. For if one applies $\alpha+\beta$ derivatives to the convolution, then it should also be true that the derivatives can be split between the two functions up to their "maximum differentiability capacity", i.e.

$$\partial^{\alpha+\beta}(f * g) = (\partial^\alpha f) * (\partial^\beta g).$$

Establishing these ideas rigorously is the content of the following proposition, whose hypotheses are basically just so that the Leibniz rule - the exchange between the derivative and the integral - can be applied.

Proposition 1.4. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in C^k(\mathbb{R}^n)$, for some $k \in \mathbb{N}$, such that $\partial^\alpha g$ is bounded for all $|\alpha| \leq k$. Then, $f * g \in C^k(\mathbb{R}^n)$ and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for all $|\alpha| \leq k$.*

Proof. For the given hypotheses, the integrals

$$\int_{\mathbb{R}^n} \partial^\alpha g(x-y) f(y) dy,$$

all exist for $|\alpha| \leq k$ and $x \in \mathbb{R}^n$; $g(x-y)f(y) \in C^k(\mathbb{R}^n)$ in the x variable; $\partial_x^\alpha\big(g(x-y)f(y)\big) = \partial^\alpha g(x-y)f(y)$ exists for all $x, y \in R^n$; and $|\partial^\alpha g(x-y)f(y)| \leq C_\alpha |f(y)| \in L^1(\mathbb{R}^n)$ independently of x.

These are the conditions required to apply the Leibniz rule, and exchange the derivative with the integral sign. (The Leibniz rule follows from a straightforward application of the Lebesgue Dominated Convergence Theorem and the Mean Value Theorem. See Theorem 2.27, pg 56 in [1], in case you need to refresh on the Leibniz rule for Lebesgue integrals). □

Before we proceed further, we need to look into the behavior of translations with respect to the L^p norm. But before we do, an important density result is required.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be open. Then, the set $C_c(\Omega)$ of continuous functions, with compact support in Ω, is dense in $L^p(\Omega)$, for all $1 \leq p < \infty$.*

Proof. We know, from one of the previous lessons, that the set of simple functions in Ω, that vanish outside of a set of finite measure, is dense in $L^p(\Omega)$, for all $1 \leq p < \infty$. So it suffices to show that, for any characteristic function χ_E of a set of finite measure $E \subset \Omega$, $\mu(E) < \infty$, one can pick a function in $C_c(\Omega)$ which is arbitrarily close in the $L^p(\Omega)$ norm. Due to the regularity of the Lebesgue measure in $\mathbb{R}^n$, for any $\varepsilon > 0$ we can choose a compact $K \subset E$ and an open O such that $E \subset O \subset \Omega$, for which $\mu(O \setminus K) < \varepsilon$. Then, using the Urysohn lemma from general topology, a continuous function of compact support can be picked $f \in C_c(\Omega)$ with $\chi_K \leq f \leq \chi_O$ and it satisfies

$$\|\chi_E - f\|_{L^p(\Omega)} \leq \mu(O \setminus K)^{1/p} < \varepsilon^{1/p}.$$

□

By using the convolution and the regularity result of Proposition 1.4 we will be able to do even better later on, and show that not only are continuous functions of compact support dense in $L^p(\Omega)$, but also the smooth functions of compact support $C_c^\infty(\Omega)$. However, the density of continuous only functions is already a fundamental tool that allows us to shift arguments from the possibly badly behaved L^p functions, to much better behaved continuous functions which are arbitrarily close.

Keep in mind that in L^∞ this type of density results with compactly supported functions never holds. In fact, for continuous functions the L^∞ norm is the same as the supremum norm. And thus, L^∞ limits of sequences of continuous functions are exactly the same as uniform limits. However, from advanced calculus, we know that uniform limits of continuous functions are continuous as well, so a discontinous function in L^∞ could never be approximated in the L^∞ norm by sequences of continuous functions: the subspace of continuous functions in L^∞ is a closed, strictly smaller, subset.

Another related ingredient - in the sense that it also uses the fact that for continuous functions the L^∞ norm is the same as supremum - that will come in handy for our toolkit of useful properties is the following result, also from advanced calculus.

Proposition 1.6. *Let $f \in C_c(\mathbb{R}^n)$. Then f is uniformly continuous, which can be written, in terms of the supremum, or L^∞ norm, as*

$$\lim_{z\to 0} \|\tau_z f(\cdot) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)} = \lim_{z\to 0} \|f(\cdot - z) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Proof. Recall the Heine-Cantor theorem from advanced calculus, that every continuous function defined on a compact set is uniformly continuous on that set, and fill in the remaining easy details. Or see the full proof in Lemma 8.4, pg. 238, of [1]. □

We now have all that is required to establish the proof of the continuity of the translation operator, in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

Theorem 1.7. *Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Then*

$$\lim_{z\to 0} \|\tau_z f - f\|_{L^p(\mathbb{R}^n)} = \lim_{z\to 0} \|f(\cdot - z) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0.$$

and we conclude that the translation of an L^p function f, $z \in \mathbb{R}^n \mapsto \tau_z f \in L^p(\mathbb{R}^n)$ is uniformly continuous in z.

Proof. The proof will follow by approximating the function $f \in L^p$ by a continuous function of compact support, for which we already know, from the preceding proposition, that the translation is continuous in the L^∞ norm.

So, given any $\varepsilon > 0$, let us start by first picking a function $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_{L^p(\mathbb{R}^n)} < \varepsilon/3$. Then, for all $|z| \le 1$ the supports of the translations $\tau_z g$ all live within a common compact set K, so that

$$\int_K |\tau_z g(x) - g(x)|^p dx \le \|\tau_z g(\cdot) - g(\cdot)\|^p_{L^\infty(\mathbb{R}^n)} \mu(K),$$

which converges to 0, as $z \to 0$, from the uniform continuity of g established in the previous proposition. Therefore $\lim_{z\to 0} \|\tau_z g - g\|_{L^p(\mathbb{R}^n)} = 0$ and we finally just have to put the approximation to f, by g, together to obtain

$$\begin{aligned}\|\tau_z f - f\|_{L^p(\mathbb{R}^n)} &\le \|\tau_z f - \tau_z g\|_{L^p(\mathbb{R}^n)} + \|\tau_z g - g\|_{L^p(\mathbb{R}^n)} + \|g - f\|_{L^p(\mathbb{R}^n)} = \\ &= \|\tau_z (g - f)\|_{L^p(\mathbb{R}^n)} + \|g - f\|_{L^p(\mathbb{R}^n)} + \|\tau_z g - g\|_{L^p(\mathbb{R}^n)} = \\ &= 2\|g - f\|_{L^p(\mathbb{R}^n)} + \|\tau_z g - g\|_{L^p(\mathbb{R}^n)} < \frac{2\varepsilon}{3} + \|\tau_z g - g\|_{L^p(\mathbb{R}^n)} < \varepsilon,\end{aligned}$$

by making z sufficiently small, where, in the equality, we used the invariance of the L^p norm under translations to conclude that $\|\tau_z (g - f)\|_{L^p(\mathbb{R}^n)} = \|g - f\|_{L^p(\mathbb{R}^n)}$.

This proves the continuity of the translation map $z \in \mathbb{R}^n \mapsto \tau_z f \in L^p(\mathbb{R}^n)$ at $z = 0$. But, again due to the invariance of the L^p norm under translations, we have $\|\tau_{z+y} f - \tau_y f\|_{L^p(\mathbb{R}^n)} = \|\tau_z f - f\|_{L^p(\mathbb{R}^n)}$ for all $z, y \in \mathbb{R}^n$, so that continuity at $y = 0$ is the same as the continuity at any other $y \in \mathbb{R}^n$, yielding the uniform continuity. □

Having established the continuity of the translation in $L^p(\mathbb{R}^n)$, for $1 \leq p < \infty$, we can go back to the convolution and obtain another family of L^p estimates, this time based on Hölder's inequality.

Theorem 1.8. *Let $1 \leq p, q \leq \infty$ be conjugate exponents and $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. Then*

1. *$f * g(x)$ is well defined for all $x \in \mathbb{R}^n$,*
2. *$f * g \in L^\infty(\mathbb{R}^n)$ and $\|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$,*
3. *$f * g$ is uniformly continuous,*
4. *If $1 < p < \infty$ ($\Leftrightarrow 1 < q < \infty$) then $\lim_{x \to \infty} f * g(x) = 0$.*

Proof. Properties (1) and (2) are a straightforward application of Hölder's inequality to the convolution integral (1.1).

Property (3) follows from the uniform continuity of the translation in L^p spaces, for either p or q must be finite. Suppose, without loss of generality, that it is $1 \leq p < \infty$ (otherwise change the roles of f and g). Then, from the previous properties of the convolution and the translation operator, we have

$$\begin{aligned} \|\tau_z(f * g) - f * g\|_{L^\infty(\mathbb{R}^n)} &= \|(\tau_z f) * g - f * g\|_{L^\infty(\mathbb{R}^n)} = \\ &= \|(\tau_z f - f) * g\|_{L^\infty(\mathbb{R}^n)} \leq \|\tau_z f - f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{when} \quad z \to 0. \end{aligned}$$

Finally, for (4), choose sequences $\{f_n\} \in C_c(\mathbb{R}^n)$ and $\{g_n\} \in C_c(\mathbb{R}^n)$ such that $f_n \to f$ in $L^p(\mathbb{R}^n)$ and $g_n \to g$ in $L^q(\mathbb{R}^n)$. Of course this can only be done if neither p or q is infinity, and that is the reason for the restriction to finite exponents only. Then, $f_n * g_n$ have compact support, due to property (5) in Theorem 1.1 and are uniformly continuous due to the previous properties of the current theorem. So $f_n * g_n \in C_c(\mathbb{R}^n)$ and $\lim_{x \to \infty} f_n * g_n(x) = 0$. And approximating $f * g$ by $f_n * g_n$ uniformly we obtain

$$\begin{aligned} \|f * g - f_n * g_n\|_{L^\infty(\mathbb{R}^n)} &\leq \|f * g - f * g_n\|_{L^\infty(\mathbb{R}^n)} + \|f * g_n - f_n * g_n\|_{L^\infty(\mathbb{R}^n)} \leq \\ &\leq \|f\|_{L^p(\mathbb{R}^n)} \|g - g_n\|_{L^q(\mathbb{R}^n)} + \|f - f_n\|_{L^p(\mathbb{R}^n)} \|g_n\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{1.2}$$

And thus, due to this uniform convergence of $f_n * g_n$ to $f * g$, the limit $\lim_{x \to \infty} f * g(x)$ must also be zero. □

To finish this lesson we observe, summarizing, that we have obtained two independent sets of L^p estimates for the convolution. On the one hand, from Young's inequality, we have

- $f \in L^p(\mathbb{R}^n), g \in L^1(\mathbb{R}^n)$ then $\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$

and on the other hand, from Hölder's inequality

- $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$ then $\|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$

This is a paradigmatic example of a case where interpolation of operators in L^p spaces will be greatly useful: by fixing $f \in L^p$, we will be able to interpolate the estimates obtained from Young, at the level of $g \in L^1$, and the ones obtained from Hölder, at the level of $g \in L^q$, to obtain convolution estimates for $g \in L^s$, with $1 \leq s \leq q$, yielding convolutions in L^r, with $p \leq r \leq \infty$. Again, as mentioned before, this will be the subject of the Riesz-Thorin interpolation theorem, to be covered in a future lesson.

For this particular case, of the interpolated convolution estimates, the result can actually be obtained by a clever and delicate use of Hölder's inequality. So we will finish with the statement of the full interpolated estimates for convolution products, although we will leave our proof of it to the application of the Riesz-Thorin interpolation theorem.

Theorem 1.9. (Generalized Young's Inequality) *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^s(\mathbb{R}^n)$ with $\frac{1}{p}+\frac{1}{s}=1+\frac{1}{r}$, $1 \leq p, s \leq r \leq \infty$. Then $f * g(x)$ is well defined by the integral* (1.1) *a.e. $x \in \mathbb{R}^n$, it is a function in $L^r(\mathbb{R}^n)$ and $\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^s(\mathbb{R}^n)}$.*

Proof. Try, as an exercise, to prove it yourself by using the generalized Hölder inequality to show first that

$$|f * g(x)|^r \leq \|f\|_{L^p}^{r-p}\|g\|_{L^s}^{r-s} \int |f(y)|^p |g(x-y)|^s dy,$$

when $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{s}$, $1 \leq p, s \leq r < \infty$, and from here obtain the final estimate. If you give up and want to see the proof you can find it in Grafakos' book [2], where it is done this way, without the Riesz-Thorin interpolation theorem, in Theorem 1.2.12, pg. 22. □

References

[1] Gerald B. Folland, *Real Analysis, Modern Techniques and Applications,* 2nd Edition, John Wiley & Sons, 1999.
[2] Loukas Grafakos, *Classical Fourier Analysis,* 3rd Edition, Sp ringer, Graduate Texts in Mathematics 249, 2014.